Accretion equations

Pablo Villanueva Domingo

April 8, 2019

Contents

1 General equations

Equations for continuity, moment conservation, energy conservation and ionization:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0 \tag{1.1}$$

$$\frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v} = -\frac{1}{\rho} \nabla P - \nabla \Phi \tag{1.2}$$

$$\frac{\partial \mathcal{E}}{\partial t} + \vec{v} \cdot \nabla \mathcal{E} + (\mathcal{E} + P) \nabla \cdot \vec{v} = \mathcal{H} \tag{1.3}$$

$$\frac{\partial n_e}{\partial t} + \nabla \cdot (n_e \vec{v}) = \Gamma n_{HI} - \alpha \, n_e^2 \tag{1.4}$$

Defining $x_e = n_e/n_H$,

$$\left(\frac{\partial}{\partial t} + \vec{v} \cdot \nabla \right) x_e = \Gamma (1 - x_e) - \alpha n_H x_e^2 \tag{1.5}$$

Writting the energy density $\mathcal{E} = \frac{3}{2} n T$ and $P = nT$, where $\rho = m_p n_H + m_e n_e \simeq m_p n_H$ and $n = n_H + n_e = n_H(1 + x_e)$, we get

$$\frac{\partial \mathcal{E}}{\partial t}+\vec{v}\cdot\nabla\mathcal{E}+(\mathcal{E}+P)\nabla\cdot\vec{v}=\left(\frac{\partial}{\partial t}+\vec{v}\cdot\nabla\right)(\frac{3}{2}nT)+\frac{5}{2}nT\nabla\cdot\vec{v} \tag{1.6}$$

$$=\frac{3}{2}n\left(\frac{\partial}{\partial t}+\vec{v}\cdot\nabla\right)T+\frac{3}{2}Tn_H\left(\frac{\partial}{\partial t}+\vec{v}\cdot\nabla\right)x_e-\frac{nT}{\rho}\left(\frac{\partial}{\partial t}+\vec{v}\cdot\nabla\right)\rho \tag{1.7}$$

$$=\frac{3}{2}nT\left[\frac{1}{T}\left(\frac{\partial}{\partial t}+\vec{v}\cdot\nabla\right)T+\frac{1}{1+x_e}\left(\frac{\partial}{\partial t}+\vec{v}\cdot\nabla\right)x_e-\frac{2}{3\rho}\left(\frac{\partial}{\partial t}+\vec{v}\cdot\nabla\right)\rho\right] \tag{1.8}$$

$$=\frac{3}{2}nT\left(\frac{\partial}{\partial t}+\vec{v}\cdot\nabla\right)ln\left(\frac{T(1+x_e)}{\rho^{2/3}}\right), \tag{1.9}$$

where in the second line, the continuity equation has been used. Therefore, we can write

$$\left(\frac{\partial}{\partial t}+\vec{v}\cdot\nabla\right)ln\left(\frac{T(1+x_e)}{\rho^{2/3}}\right)=\frac{2}{3nT}\mathcal{H} \tag{1.10}$$

Expanding universe:

$$\frac{\partial\rho}{\partial t}+3H\rho+\frac{1}{a}\nabla\cdot(\rho\vec{v})=0 \tag{1.11}$$

$$\frac{\partial\vec{v}}{\partial t}+H\vec{v}+\vec{v}\cdot\frac{1}{a}\nabla\vec{v}=-\frac{1}{\rho}\frac{1}{a}\nabla P-\frac{1}{a}\nabla\Phi \tag{1.12}$$

$$\frac{\partial\mathcal{E}}{\partial t}+\vec{v}\cdot\frac{1}{a}\nabla\mathcal{E}+(\mathcal{E}+P)(3H+\nabla\cdot\vec{v})=\mathcal{H} \tag{1.13}$$

$$\left(\frac{\partial}{\partial t}+\vec{v}\cdot\frac{1}{a}\nabla\right)x_e=\Gamma(1-x_e)-\alpha n_H x_e^2 \tag{1.14}$$

$$\left(\frac{\partial}{\partial t}+\vec{v}\cdot\frac{1}{a}\nabla\right)ln\left(\frac{T(1+x_e)}{\rho^{2/3}}\right)=\frac{2}{3nT}\mathcal{H} \tag{1.15}$$

Should we include the a factors in our steady state approximation, or would it overcounting the expanding-universe effect (since it is included in the boundary term...)?

2 Spherical steady state case

For the spherical symmetric steady state case:

$$4\pi r^2\rho|v|=\dot{M}=constant \tag{2.1}$$

$$v\frac{dv}{dr}=-\frac{1}{\rho}\frac{dP}{dr}-\frac{GM}{r^2} \tag{2.2}$$

$$\frac{v\rho^{2/3}}{(1+x_e)}\frac{d}{dr}\left(\frac{T(1+x_e)}{\rho^{2/3}}\right)=\frac{2}{3n_H(1+x_e)}\mathcal{H} \tag{2.3}$$

$$v\frac{dx_e}{dr}=\Gamma(1-x_e)-\alpha n_H x_e^2 \tag{2.4}$$

where we have to use $P=nT=\rho(1+x_e)T/m_H$. The first equation is equivalent to

$$\frac{1}{\rho}\frac{\partial\rho}{\partial r}+\frac{1}{v}\frac{\partial v}{\partial r}+\frac{2}{r}=0 \tag{2.5}$$

We rewrite the system of equations in an useful way to solve numerically, leaving the radial derivatives at the left side:

$$v = -\frac{\dot{M}}{4\pi r^2 \rho} \tag{2.6}$$

$$\frac{d\rho}{dr} = \frac{\rho}{\frac{5}{3}\frac{T(1+x_e)}{m} - v^2}\left(\frac{2v^2}{r} - \frac{GM}{r^2} - \frac{2}{3}\frac{\mathcal{H}}{\rho v}\right) \tag{2.7}$$

$$\frac{dT}{dr} = \frac{2}{3}T\frac{1}{\rho}\frac{d\rho}{dr} - T\frac{1}{1+x_e}\frac{dx_e}{dr} + \frac{2}{3}\frac{\mathcal{H}}{nv} \tag{2.8}$$

$$\frac{dx_e}{dr} = \frac{1}{v}\left(\Gamma(1-x_e) - \alpha n_H x_e^2\right) \tag{2.9}$$

Defining the Bondi scales:

$$v_B = \sqrt{\frac{P_\infty}{\rho_\infty}} = \sqrt{\frac{T_\infty}{m_H}} \simeq 1.5 \times 10^4 cm/s \sqrt{\frac{T_\infty(1+z)}{T_{CMB}(z)}}, \tag{2.10}$$

$$r_B = \frac{GM}{v_B^2} \simeq 1.9 \times 10^{-4} kpc \frac{M}{M_\odot}\frac{T_{CMB}(z)}{T_\infty(1+z)}, \tag{2.11}$$

$$t_B = \frac{r_B}{v_B} = \frac{GM}{v_B^3} \simeq 3.9 \times 10^{13} s \frac{M}{M_\odot}\left(\frac{T_{CMB}(z)}{T_\infty(1+z)}\right)^{3/2}. \tag{2.12}$$

Assuming $T_\infty = Tad(z)$, we write the above formulas as:

$$v_B \simeq 3.67 \times 10^4 cm/s \left(\frac{1+z}{30}\right), \tag{2.13}$$

$$r_B \simeq 3.18 \times 10^{-5} kpc \frac{M}{M_\odot}\left(\frac{30}{1+z}\right)^2, \tag{2.14}$$

$$t_B = \frac{r_B}{v_B} \simeq 2.68 \times 10^{12} s \frac{M}{M_\odot}\left(\frac{30}{1+z}\right)^3. \tag{2.15}$$

Using the dimensionless variables $u = v/v_B$, $\hat{\rho} = \rho/\rho_\infty$, $\hat{T} = T/T_\infty$, and defining also

$$\lambda = \frac{\dot{M}}{4\pi r_B^2 v_B \rho_\infty}, \tag{2.16}$$

we write the set of equations in a dimensionless form:

$$u = -\frac{\lambda}{x^2 \hat{\rho}} \tag{2.17}$$

$$\frac{d\hat{\rho}}{dx} = \frac{\hat{\rho}}{\frac{5}{3}\hat{T}(1+x_e) - u^2}\left(\frac{2u^2}{x} - \frac{1}{x^2} - \frac{2}{3}\frac{\hat{\mathcal{H}}}{u}\right) \tag{2.18}$$

$$\frac{d\hat{T}}{dx} = \frac{2}{3}\hat{T}\frac{1}{\hat{\rho}}\frac{d\hat{\rho}}{dx} - \hat{T}\frac{1}{1+x_e}\frac{dx_e}{dx} + \frac{2}{3}\frac{\hat{\mathcal{H}}}{u(1+x_e)} \tag{2.19}$$

$$\frac{dx_e}{dx} = \frac{t_B}{u}\left(\Gamma(1-x_e) - \alpha n_H x_e^2\right) \tag{2.20}$$

where we have defined the dimensionless heating term:

$$\hat{\mathcal{H}} = \frac{\mathcal{H}\, t_B}{n_H T_\infty} \tag{2.21}$$

In an expanding universe:

$$u = -\frac{\lambda}{x^2 \hat{\rho}} \tag{2.22}$$

$$\frac{d\hat{\rho}}{dx} = \frac{\hat{\rho}}{\frac{5}{3}\hat{T}(1+x_e) - u^2}\left(\frac{2u^2}{x} - \frac{1}{ax^2} - \frac{2}{3}\frac{\hat{\mathcal{H}}a}{u}\right) \tag{2.23}$$

$$\frac{d\hat{T}}{dx} = \frac{2}{3}\hat{T}\frac{1}{\hat{\rho}}\frac{d\hat{\rho}}{dx} - \hat{T}\frac{1}{1+x_e}\frac{dx_e}{dx} + \frac{2}{3}\frac{\hat{\mathcal{H}}a}{u(1+x_e)} \tag{2.24}$$

$$\frac{dx_e}{dx} = \frac{at_B}{u}\left(\Gamma(1-x_e) - \alpha n_H x_e^2\right) \tag{2.25}$$

3 Radiation and Heating

3.1 Compton cooling

The Compton cooling term is given by

$$\mathcal{H}_{Compton} = \frac{4\sigma_T \rho_\gamma n_H x_e}{m_e c}(T_\gamma - T) \tag{3.1}$$

Therefore, we rewrite it in a dimensionless way:

$$\hat{\mathcal{H}}_{Compton} = \frac{t_B}{t_C}(\hat{T}_\gamma - \hat{T}), \tag{3.2}$$

where $t_C^{-1} = 4\sigma_T \rho_\gamma x_e/(m_e c)$. $t_B/t_C = 1.6 \times 10^{-6}(1+z)^{5/2} x_e \frac{M}{M_\odot}$ for $T_\infty = T_\gamma$. For $T_\infty = T_{ad}(z)$, $t_B/t_C \simeq 9 \times 10^{-2}((1+z)/30) x_e M/M_\odot$.

3.2 Adiabatic cooling

The adiabatic cooling term can be written as

$$\mathcal{H}_{Ad} = 3nH(z)T \tag{3.3}$$

so

$$\frac{2}{3}\frac{\mathcal{H}_{Ad}}{n} = 2H(z)T \tag{3.4}$$

and

$$\frac{2}{3}\hat{\mathcal{H}}_{Ad} = 2\frac{t_B}{t_H}\hat{T}(1+x_e) \tag{3.5}$$

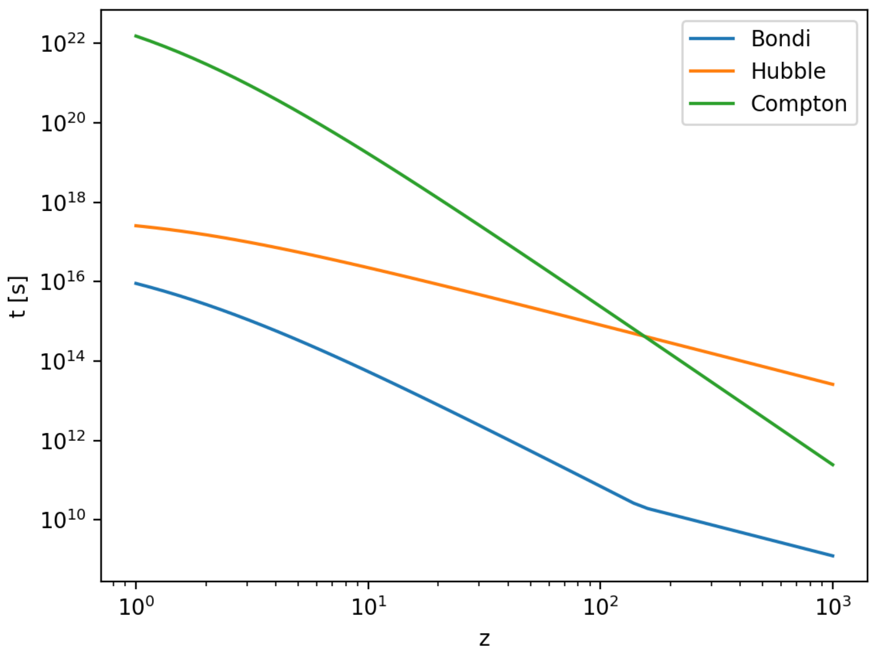

Figure 1: Relevant timescales, assuming $x_e = 10^{-4}$ and $M = M_\odot$.

3.3 Heating by BH

We consider a radiation flux produced by a point source:

$$J(E) = \frac{\mathcal{L}(E)}{4\pi r^2} e^{-\tau(E)} \tag{3.6}$$

where J/c is the energy density per unit of energy, with units $[J/c] = L^{-3}$, with

$$\tau(E) = \int_0^r dr n_H \sigma(E) x_{HI} \tag{3.7}$$

We can compute the ionizing and heating terms:

$$\Gamma = \int_{E_0}^{\infty} \frac{dE}{E} \sigma(E) J(E) (1 + \frac{E - E_{th}}{E_{th}} f_{ion}(E, x_e)) \tag{3.8}$$

$$\mathcal{H}_{ph} = n_H x_{HI} f_{heat}(x_e) \int_{E_0}^{\infty} \frac{dE}{E} \sigma(E) J(E) (E - E_{th}) \tag{3.9}$$

where $x_{HI} = 1 - x_e$. Writing the luminosity function as $\mathcal{L}(E) = L\psi(E)$ with $\int_{E_0}^{\infty} dE \psi(E) = 1$ and $\sigma(E) \simeq \sigma_0 (E/E_0)^{-p}$, with $\sigma_0 \simeq 4.25 \times 10^{-21} cm^2$ and $p \simeq 3$

$$\hat{\mathcal{H}}_{ph} = \frac{L}{T_\infty / t_B} \frac{\sigma_0}{r_B^2} \frac{1}{4\pi x^2} x_{HI} f_{heat}(x_e) \int_{E_0}^{\infty} \frac{dE}{E} \left(\frac{E_0}{E} \right)^p \psi(E) e^{-\tau(E)} (E - E_{th}) \tag{3.10}$$

Assuming an Eddington ratio $\xi = L/L_{Edd}$, for $T_\infty = T_\gamma$, $\frac{L}{T_\infty/t_B} = 1.3 \times 10^{67} (\frac{M}{M_\odot})^2 / (1+z)^{3/2}$. For $T_\infty = T_{ad}$, at z=30, $\frac{L}{T_\infty/t_B} = 1.3 \times 10^{65} \xi (\frac{M}{M_\odot})^2$ and $\sigma_0/(4\pi r_B^2) = 3.9 \times 10^{-56} (\frac{M}{M_\odot})^{-2}$, so

$\frac{L}{T_\infty/t_B}\sigma_0/(4\pi r_B^2) \simeq 5{\times}10^9\xi$, independent of mass. Assuming $\psi = AE^{-1}$ with $A = (log(E_2/E_1))^{-1} \simeq 0.16$, we can perform the energy integral:

$$I/A \simeq \int_{E_1}^{E_2} \frac{dE}{E}\left(\frac{E_0}{E}\right)^p \frac{1}{E} e^{-\tau_0\left(\frac{E_0}{E}\right)^p}(E - E_{th}) \tag{3.11}$$

$$\simeq \int_{E_1}^{E_2} \frac{dE}{E}\left(\frac{E_0}{E}\right)^p e^{-\tau_0\left(\frac{E_0}{E}\right)^p} = \int_1^{E_2/E_1} \frac{dy}{y} y^{-p} e^{-\tau_0 y^{-p}} \tag{3.12}$$

$$= \frac{(-1)}{p\tau_0}\int_1^{\tau_0(E_2/E_1)^{-p}} dwe^{-w} = \frac{1}{p\tau_0}(e^{-\tau_0(E_2/E_1)^{-p}} - e^{-\tau_0}) = \frac{e^{-\tau_0}}{p\tau_0}\left(e^{-\tau_0((E_1/E_2)^p-1)} - 1\right) \tag{3.13}$$

For low tau, we get $I/A \simeq \frac{1}{p}\left(1 - \left(\frac{E_1}{E_2}\right)^p\right)$. For low tau, we set it to zero, and therefore

$$I/A \simeq \int_{E_1}^{E_2} \frac{dE}{E}\left(\frac{E_0}{E}\right)^p = \int_1^{E_2/E_1} dyy^{-p-1} = \frac{1}{p}\left(1 - \left(\frac{E_1}{E_2}\right)^p\right) \simeq \frac{1}{p}, \tag{3.14}$$

so $I/A \sim 1/3 \sim 0.3$. However, taking into account a more realistic function for the cross section as a broken power law, $I/A \simeq 0.6$. Assuming also $f_{heat} \simeq 0.15$ (which stands for a neutral medium), we finally get

$$\hat{\mathcal{H}}_{ph} \sim 5.4 \times 10^7 \xi \frac{1}{x^2} x_{HI}. \tag{3.15}$$

The Eddington ratio can be written as

$$\xi = \frac{L}{L_{Edd}} = \frac{L}{\dot{M}c^2}\frac{\dot{M}c^2}{L_{Edd}} = \epsilon\lambda\, t_B(T_\infty, z)\, n_H(z) c\sigma_T \simeq 2.7 \times 10^{-4}\epsilon\lambda\frac{M}{M_\odot} \tag{3.16}$$

where $\epsilon = L/(\dot{M}c^2)$. The last equality stands for $T_\infty = T_{ad}$. Assuming $\lambda \sim 0.01$ and treating ϵ as in Poulin et al., we get $\xi \simeq 6.6 \times 10^{-10}$. Then $\hat{\mathcal{H}}_{ph} \sim 3.4 \times 10^{-2}\frac{1}{x^2}x_{HI}$. Comparing to $\hat{\mathcal{H}}_{Compton} \sim 9 \times 10^{-2} x_e M/M_\odot(\hat{T}_\gamma - 1) \sim 5 \times 10^{-5} M/M_\odot$, heating is dominant over Compton cooling for $x < \sqrt{2M_\odot/M}$ (check these numbers).