Notes on accretion disk models

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1 Accretion equations

See reviews: Yuan-Narayan 2014 [1], Abramowicz-Fragile 2013 [2], Lasota 2015 [3], NDAF review

Angular velocity $\Omega = v_\phi/r$, Keplerian angular velocity: $\Omega_K^2 = GM/r^3$, $v_K = \Omega_K r$. $l = r^2\Omega = rv$ is the specific angular momentum. If rigid rotation, $l \propto r^2$, if Keplerian, $l = \sqrt{GMr}$, constant angular momentum $l = constant$ implies $\Omega \propto r^{-2}$. Pressure $P = c_s^2\rho$, surface density $\Sigma = \int_{-H}^{+H} \rho dz \simeq 2\rho H$. Consider that only ρ varies with z. The only stress tensor component is $\tau_{r\phi} = -\nu\rho\sigma_{r\phi}$, with ν the viscosity and the shear $\sigma_{r\phi} = r\partial_r\Omega$[1].

$$\partial_r(rv\rho) = 0 \tag{1.1}$$

$$v\partial_r v - r(\Omega^2 - \Omega_K^2) = -\frac{1}{\rho}\partial_r P \tag{1.2}$$

$$\rho v\partial_r(r^2\Omega) = \frac{1}{r}\partial_r\left(\nu\rho r^3\partial_r\Omega\right) \tag{1.3}$$

$$\frac{1}{\rho}\partial_z P = -\frac{GMz}{r^3} \tag{1.4}$$

$$\rho v\left(\partial_r e - \frac{P}{\rho^2}\partial_r\rho\right) = \nu\rho r^2\left(\partial_r\Omega\right)^2 - q_- = q_+ - q_-, \tag{1.5}$$

with $q_+ = \nu\rho r^2\left(\partial_r\Omega\right)^2$ the heating via viscous dissipation and q_- the cooling processes. Writing the left hand side of the above equation as $q_{adv} = \rho vT\partial_r s = \rho v\left(\partial_r e - \frac{P}{\rho^2}\partial_r\rho\right)$, the advective cooling rate, we can write the above equation as

$$q_{adv} = q_+ - q_- = fq_+ \tag{1.6}$$

$f = q_{adv}/q_+ = 1-(q_-/q_+)$ measures the relative importance of advection. Out of the total heat energy q_+ released by viscous dissipation per unit volume per unit time, a fraction f is advected and the rest $(1-f)$ is radiated.

One can write $P \propto \rho^\gamma$, $P = c_s^2\rho$ and $e = c_s^2/(\gamma-1)$.

Eq. 1.1 can be integrated as

$$\dot{M} = -2\pi\Sigma rv \tag{1.7}$$

[1]Other components of the stress tensor ca be present, but vanishes after averaging the fluid equations over z and ϕ, see Gordon Ogilvie lectures.

	Optically thick ($\tau \gg 1$)	Optically thin ($\tau \ll 1$)
Radiative dominated ($q_+ = q_-$)	SSD	SLE
Advective dominated ($q_{adv} = q_+$)	Slim disk	ADAF

Table 1: Accretion scenarios

Solving Eq. 1.4 for $P = c_s^2 \rho$, we get

$$\rho = \rho_0 \exp\left(-\frac{\Omega_K^2}{2c_s^2} z^2\right) \tag{1.8}$$

We thus identify $H \simeq c_s/\Omega_K$.

Employing Eq. 1.7, we can integrate Eq. 1.3 as

$$\Sigma v \partial_r (r^2 \Omega) = \frac{1}{r} \partial_r \left(\nu \Sigma r^3 \partial_r \Omega\right) \quad \Rightarrow \quad -\frac{\dot{M}}{2\pi} r^2 \Omega = \nu \Sigma r^3 \partial_r \Omega + K \tag{1.9}$$

with K a constant, related with the angular momentum at some radius j_0. The above equation can be written as

$$\dot{M}\left(r^2 \Omega - j_0\right) = -2\pi \nu \Sigma r^3 \partial_r \Omega \tag{1.10}$$

Shakura-Sunyaev viscosity prescription:

$$\tau_{r\phi} = -\nu \rho \sigma_{r\phi} = -\alpha P, \quad \nu = \alpha c_s H \simeq \alpha \frac{c_s^2}{\Omega_K} \tag{1.11}$$

or $\frac{2}{3}\alpha$ instead of α.

Radiative efficiency:

$$\epsilon = \frac{L}{\dot{M} c^2} \tag{1.12}$$

When $\dot{M}$ is very low, the gas density ρ is also low and the radiative cooling rate q_- (which decreases rapidly with decreasing ρ) becomes negligibly small. The viscous heating rate is then balanced primarily by energy advection rather than cooling, and $f \simeq 1$. Increasing $\dot{M}$, there is a moment at $\dot{M}_{ADAF}$ when $f = 0$. Thus, ADAF works for lower $\dot{M}$. Between $\dot{M}_{ADAF}$ and $\dot{M}_{LHAF}$, $q_c + q_+ = q_-$, with q_c the compression cooling, and $f < 1$.

Also

$$q_{adv} = \frac{1}{2} f \alpha \Sigma c_s (r \partial_r \Omega)^2 \simeq f \alpha \frac{P}{\Omega_K} (r \partial_r \Omega)^2 \tag{1.13}$$

Optical depth:

$$\tau = \Sigma H \kappa \tag{1.14}$$

with κ the opacity coefficient. Large Σ implies optically thick mediums. Optically thick means that photons thermalize and the emitted spectrum is black body. Otherwise, it is dominated by other non-thermal processes, such as synchrotron.

2 Overview of accretion scenarios

Cold flows (temperature lower than the virial temperature), optically thick:

- SSD, or Thin disk: geometrically thin ($H \ll r$), optically thick, Keplerianly rotating $\Omega = \Omega_K$, viscous heating is balanced by the radiative cooling, $q_+ = q_-$ ($f = 0$). Applies to X-ray binaries and bright AGNs. Sub-Eddington rates, luminosity efficiency of approximately $\epsilon \sim 10\%$.

- Slim disk: geometrically slim (but no thin) ($H \lesssim r$), optically thick. Super-Eddington rates. Most of the photons are carried by the accretion flow and finally fall into the BH: the main cooling mechanism is advection rather than radiation. Advection dominated due to long radiative diffusion time, unlike ADAF, which is advection dominated due to long cooling time. Therefore, it does not radiates photons efficiently, $\epsilon < 0.1$. $f = 1$.

- NDAF: mostly same than slim disk, but neutrino cooling dominates

- Thick disk (polish doughnouts): $H > r$, high accretion rate. Is it NDAF? Radiatively inefficient, $\epsilon \ll 0.1$. Only gravity and advection, does not present radiation nor viscosity. Simplest accretion disk. f not defined, I think. l constant at the center, Keplerian at the outer parts.

Hot flows (temperature close to the virial temperature), optically thin:

- SLE (Shapiro et al 1976): hot accretion flow, optically thin, two temperatures. But thermally unstable, so is unlikely to be realized in nature. $q_+ = q_-$ ($f = 0$).

- ADAF: geometrically thick ($H \sim r$), optically thin. Extremely high temperature. The main cooling mechanism is also advection rather than radiation. $\epsilon \ll 0.1$. Viscously dissipates accreted energy, which goes into heating. It applies e.g. to Sgr A*. $f = 1$.

- LHAF (luminous hot accretion flow): an extension of an ADAF to accretion rates above the original range of validity of the ADAF solution, leading to high efficiencies and luminosities, $\epsilon \sim 0.1$. $f < 0$.

Hot accretion flows should have strong outflows and jets. Two variants of ADAF: adiabatic inflow-outflow solution (ADIOS) and convection-dominated accretion flow (CDAF), which emphasize the roles of two distinct physical phenomena in hot accretion flows: outflows and convection. Also electron ADAF (eADAF), with even dimmer luminosities, because even electrons are also advective dominated.

The reason ADAFs are advection-dominated is that the accreting gas has a low density (because of the low mass accretion rate) and the thermal structure of the plasma is two-temperature. Because of the low density, very little of the heat energy in the ions gets transferred to the electrons through Coulomb collisions. Since the ions hardly radiate at all, they retain their thermal energy and advect essentially all of it to the center ("Why Do AGN Lighthouses Switch Off?").

3 Shakura-Sunyaev disk (SSD)

Assumptions

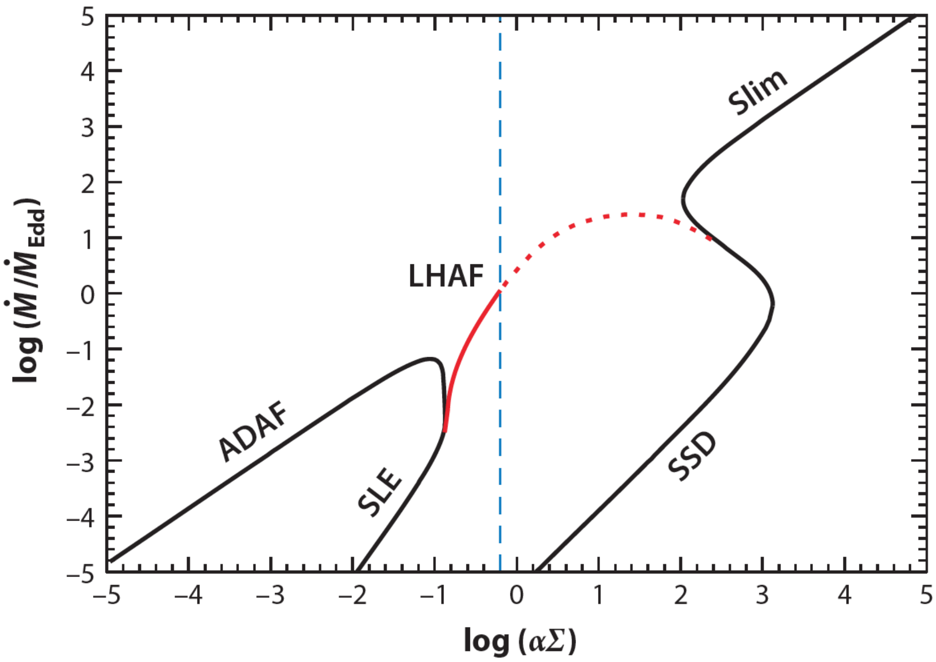

Figure 1: Accretion models, from Yuan, Narayan 2014 [1].

- Assume a thin disk. Since $H \simeq c_s/\Omega_K$, $H/r \simeq c_s/v_K$. Thus, it presents a large Mach number (in the azimuthal direction), v_K/c_s. Actually, $v \sim \nu/r \sim \alpha(H/r)c_s$ (as explicitly checked later). The orbital motion is highly supersonic while the accretion flow is highly subsonic. Being thin $H \ll r$ implies that it presents small pressure and is cold ($c_s \ll v_K$). Thus, $v, c_s \ll v_K$.
- Assume balance between viscosity heating and radiative cooling, $q_+ = q_-$. This condition is not completely independent from the above one, since $q_{adv}/q_+ \sim H/r$, negligible for a thin disk.

Since $v, c_s \ll v_K$, thus from Eq. 1.2 we have Keplerian motion, $\Omega = \Omega_K$. From Eq. 1.9, using the fact that $\Omega = \Omega_K$, we get

$$\nu\Sigma = \frac{\dot{M}}{3\pi}\left[1 - \sqrt{\frac{r_*}{r}}\right] \tag{3.1}$$

Note that $\dot{M} \propto \alpha\Sigma$. Since $q_+ = q_-$, the radiative cooling is given by

$$0 = \pi_{\mu\nu}\sigma^{\mu\nu} - \nabla\cdot\vec{F} = \nu\rho(r\partial_r\Omega)^2 - \partial_z F, \;\; \Rightarrow \;\; F = \frac{1}{2}\nu\Sigma(r\partial_r\Omega)^2 \tag{3.2}$$

and thus

$$2Hq_- = 2F = \nu\Sigma r^2\left(\partial_r\Omega\right)^2 = \frac{3GM\dot{M}}{4\pi r^3}\left[1 - \sqrt{\frac{r_*}{r}}\right] \tag{3.3}$$

and the flux (scholarpedia)

$$F = \frac{3GM\dot{M}}{8\pi r^3}\left[1 - \sqrt{\frac{r_*}{r}}\right] \tag{3.4}$$

$q_- = \nabla \cdot \vec{F} \simeq F/H$ denotes the rate at which energy is emitted per unit area. One can define an effective temperature as $F = \sigma T_{eff}^4$

On the other hand, $\vec{F} = \frac{4}{3}(\nabla \sigma T^4)/(\rho\kappa) \simeq \frac{16}{3}\sigma T^4/\tau$, with $\tau = \kappa\rho H = \Sigma\kappa/2$.

The luminosity is given by

$$L = 2\pi \int_{-H}^{H} dz \int_{r_*}^{\infty} dr\, r\, q_- = 2\pi \int_{r_*}^{\infty} dr\, \nu \Sigma r^3 \left(\partial_r \Omega\right)^2 \tag{3.5}$$

$$= \frac{3}{2} \frac{GM\dot{M}}{r_*} \int_1^{\infty} \frac{dx}{x^2} \left(1 - \sqrt{\frac{1}{x}}\right) = \frac{1}{2} \frac{GM\dot{M}}{r_*} \tag{3.6}$$

(e.g. Blaes 2002) If the Innermost Stable Circular Orbit (ISCO) radius is taken as r_*, $r_{\rm ISCO} = 3r_S$, with $r_S = 2GM/c^2$ the Schwarzschild radius, we get

$$\epsilon = \frac{L}{\dot{M}c^2} = \frac{1}{12} \sim 0.1, \tag{3.7}$$

as states the classical result. Other choices of the radius lead to 1/6 or 1/16 instead. Note that the above computation does not need to assume the ν parameterization with α.

At large radii, $\nu\Sigma \simeq \dot{M}/3\pi$, $\nu \simeq \frac{2}{3} r v$. When $H \sim r$, $v \sim \alpha c_s$.

4 ADAF

eADAF if $\dot{M} \lesssim \dot{M}_{\rm c,eADAF} \simeq 0.001\alpha^2 \dot{M}_{Edd}$, with $\dot{M}_{Edd} = 10 L_{Edd}/c^2$.

ADAF if $\dot{M}_{\rm c,eADAF} \lesssim \dot{M} \lesssim \dot{M}_{\rm c,ADAF} \simeq 0.1\alpha^2 \dot{M}_{Edd}$.

LHAF if $\dot{M}_{\rm c,ADAF} \lesssim \dot{M} \lesssim \dot{M}_{\rm c,LHAF} \simeq 0.07\alpha \dot{M}_{Edd}$

5 Abramowicz et al. 1995

Following the work of Abramowicz et al. 1995. [4], see also Lasota 2015 [3].

$$Q_{adv} = \frac{\dot{M}}{2\pi R^2} c_s^2 \xi \tag{5.1}$$

$$Q_+ = \frac{3}{4\pi} \dot{M} \tag{5.2}$$

For optically thin, we employ only bremssthralung:

$$q_- = \frac{4\sqrt{2}}{\pi^{3/2}} n_e \bar{n} \sigma_T c \alpha_{fs} m_e c^2 \left(\frac{T}{m_e c^2}\right)^{1/2} \tag{5.3}$$

or $Q_- = 2Hq_-$,

$$Q_- = \mathcal{A} H \rho^2 T^{1/2} = \frac{1}{4} \mathcal{A} \Sigma^2 T^{1/2} H^{-1}, \;\; \mathcal{A} = 1.24 \times 10^{21} erg\, cm^{-2}\, s^{-1} \tag{5.4}$$

For optically thick,

$$q_- = \frac{8\sigma T^4}{3H\tau} \tag{5.5}$$

$$\nu = \frac{2}{3}\alpha c_s H \tag{5.6}$$

$r = R/R_S$

$$\left(\frac{H}{R}\right)^2 = 2r\frac{c_s^2}{c^2} \tag{5.7}$$

$$\dot{M} = 2\pi\alpha\Sigma|v|R \tag{5.8}$$

Far enough from the inner part of the disk,

$$\nu\Sigma = \frac{\dot{M}}{3\pi} \tag{5.9}$$

$\nu = \frac{2}{3}|v|R$

$$\dot{M} = 2\pi\alpha\Sigma c_s H \tag{5.10}$$

$$\dot{m} = \sqrt{2}r^{3/2}\alpha\Sigma\kappa\frac{c_s^2}{c^2} \tag{5.11}$$

$$\frac{c_s^2}{c^2} = \frac{1}{\sqrt{2}r^{3/2}}\frac{\dot{m}}{\alpha\Sigma\kappa} \tag{5.12}$$

$$\frac{\dot{M}}{2\pi R^2}c_s^2\xi = Q_+ = \frac{3}{4\pi}\dot{M} - Q_- \tag{5.13}$$

$$\xi\dot{m}^2\frac{1}{\sqrt{2r}\alpha\kappa\Sigma} = \frac{3}{4}\dot{m} - Q_-\frac{\kappa r}{2GMc}R^2 \tag{5.14}$$

5.1 Optically thin

$$\xi\dot{m}^2 = \frac{3}{2\sqrt{2}}\sqrt{r}\alpha\kappa\Sigma\,\dot{m} - Br^2\alpha\left(\kappa\Sigma\right)^3 \tag{5.15}$$

with $B = \frac{2\sqrt{2}}{\pi^{3/2}}\sqrt{m_e/m_p}\alpha_{fs} = \frac{1}{2\sqrt{2}}\mathcal{A}\frac{\sqrt{m_p}}{\kappa c^2} = 1.22\times10^{-4}$

Solve the above equation for $\dot{m}$. Two limiting cases:

- **ADAF limit.**

  Neglect radiation cooling, $q_{adv} \simeq q_+$,

$$\dot{m} \simeq \frac{3}{2\sqrt{2}}\frac{\sqrt{r}}{\xi}\alpha\kappa\Sigma. \tag{5.16}$$

- **SLE limit.**

  Neglect advective cooling, , $q_+ \simeq q_-$,

$$\dot{m} \simeq \frac{2\sqrt{2}}{3}Br^{3/2}\left(\kappa\Sigma\right)^2. \tag{5.17}$$

Both solutions $\dot{m}_+$ merge at a critical maximum density

$$\Sigma_{max} = \frac{9}{32}\frac{\alpha}{\kappa B \xi r} \tag{5.18}$$

which implies a maximum accretion rate of

$$\dot{m}_{max} = \frac{27}{64\sqrt{2}}\frac{\alpha^2}{B\xi^2\sqrt{r}} \simeq 3.5 \times 10^3 \frac{\alpha^2}{\xi^2\sqrt{r}} \tag{5.19}$$

Equating the ADAF and SLE limits gives a $\dot{m}$ larger by a factor 4.

One should require that the Bondi accretion rate does not exceed the above maximum accretion rate in the ADAF scenario

$$\dot{M}_B = 2\pi\alpha\lambda\rho_\infty \frac{(GM)^2}{c_{s,\infty}^3} \tag{5.20}$$

or

$$\dot{m}_B = \frac{1}{2}\lambda\alpha n_{b,\infty}\sigma_T c\, t_B(M) \tag{5.21}$$

In spherical accretion, λ goes from ~ 0.1 to ~ 1 from $\gamma = 5/3$ to $\gamma = 1$.

Using the disk expressions from above, one finds a similar expression

$$\dot{m}_B = \lambda\alpha n_{b,\infty}\sigma_T c\, t_B(M) \tag{5.22}$$

At a Bondi radius, $R_B = GM/c_s^2$, $t_B = R_B/c_s = GM/c_s^3$, $r_B = R_B/R_S = c^2/(2c_s^2)$,

$$\dot{m}_{max,B} = \frac{27}{64}\frac{\alpha^2 c_s}{B\xi^2 c} \tag{5.23}$$

$\dot{m}_B \lesssim \dot{m}_{max,B}$ implies

$$\sigma_T n_\infty R_S \left(\frac{c}{c_{s,\infty}}\right)^4 \lesssim \frac{27}{32}\frac{\alpha^2}{B\xi^2} \tag{5.24}$$

$$5 \times 10^{-4}(1+z)^3 \left(\frac{1\mathrm{K}}{T_\infty}\right)^2 \left(\frac{M}{\mathrm{M}_\odot}\right)^2 \frac{\xi^2}{\alpha} \lesssim 1 \tag{5.25}$$

Note that a proper evaluation of these models requires more complex physics. ADAF and SLE require two temperatures for the ions and electrons.

5.2 Optically thick

For optically thick,

$$Q_- = \frac{16\sigma T^4}{3\kappa_R \Sigma} \tag{5.26}$$

Take $\kappa_R = \kappa_T$.

6 Self-Similar Solution

Narayan, Yi 1994 [5]

It recovers SSD for $f = 0$, and applies to ADAF for $f = 1$.

$$\epsilon' = \epsilon/f, \quad \epsilon = \frac{5/3 - \gamma}{\gamma - 1} \tag{6.1}$$

$$v \simeq -\frac{3\alpha}{5 + 2\epsilon'} v_K \tag{6.2}$$

$$\Omega^2 \simeq \frac{2\epsilon'}{5 + 2\epsilon'} \Omega_K^2 \tag{6.3}$$

$$c_s^2 \simeq \frac{2}{5 + 2\epsilon'} v_K^2 \tag{6.4}$$

and $v/c_s = -\frac{3}{2}\alpha c_s/v_K$

$$H \simeq \sqrt{\frac{2}{5 + 2\epsilon'}} r \tag{6.5}$$

$$\dot{M} = \frac{6\pi\alpha\Sigma\sqrt{GMr}}{5 + 2\epsilon'} \tag{6.6}$$

or, in dimensionless quantities,

$$\dot{m} = \frac{3}{\sqrt{2}} \frac{\alpha\Sigma}{5 + 2\epsilon'} \kappa\sqrt{\hat{r}} \tag{6.7}$$

Note that it has the same dependences than Eq. 5.16, differing only by a constant factor 5/2 (for $\epsilon = 0$ and $\xi = 1$).

$\dot{m} = \dot{M}c^2/L_{Edd}$, $\hat{r} = r/r_{Sch}$, $\kappa = \sigma_T/m_p$

$$L_{Edd} = \frac{4\pi GM m_p c}{\sigma_T} \tag{6.8}$$

7 Slim disk

Luminosity can be approximated as [6]

$$L = 2L_{Edd}\left(1 + \ln\left(\frac{1}{50}\frac{\dot{M}c^2}{L_{Edd}}\right)\right) \tag{7.1}$$

see a derivation in [3]

References

[1] F. Yuan and R. Narayan, *Hot accretion flows around black holes*, *Ann. Rev. Astron. Astrophys.* **52** (2014) 529–588, [1401.0586].

[2] M. A. Abramowicz and P. C. Fragile, *Foundations of Black Hole Accretion Disk Theory*, *Living Rev. Rel.* **16** (2013) 1, [1104.5499].

[3] J.-P. Lasota, *Black hole accretion discs*, 1505.02172.

[4] M. A. Abramowicz, X. Chen, S. Kato, J.-P. Lasota and O. Regev, *Thermal Equilibria of Accretion Disks*, ApJ **438** (Jan., 1995) L37, [astro-ph/9409018].

[5] R. Narayan and I. Yi, *Advection-dominated Accretion: A Self-similar Solution*, ApJ **428** (June, 1994) L13, [astro-ph/9403052].

[6] J. M. Wang, J. Qiu, P. Du and L. C. Ho, *Self-shadowing Effects of Slim Accretion Disks in Active Galactic Nuclei: The Diverse Appearance of the Broad-line Region*, *Astrophys. J.* **797** (2014) 65, [1410.5285].