Dynamical Microeconomics

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April 17, 2020

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1 Market dynamics

An economist is a man who states the obvious in terms of the incomprehensible.

ALFRED A. KNOPF

Consider that the consumer/seller i has n_i units of a given product. The different consumers/sellers can trade among them with this product. Consider also that there is no creation/destruction of the product, just commerce. The evolution of the number of units of the product for i is given by the **master equation**:

$$\dot{n}_i = \sum_j (C_{ij} n_j - C_{ji} n_i), \tag{1.1}$$

where C_{ij} is a matrix which gives the rate of purchases that i buy to j, per unit of product, and should depend on the money of the consumer $\$_i$, among other parameters. This matrix has $N(N-1)$ degrees of freedom, since the diagonal terms are irrelevant in the master equation (the ii term is canceled in eq. 1.1), and can be set equal to zero. This set of equations conserve the total number of unities of the product:

$$\sum_i \dot{n}_i = \sum_{ij} (C_{ij} n_j - C_{ji} n_i) = \sum_{ij} (C_{ij} n_j - C_{ij} n_j) = 0. \tag{1.2}$$

If we have several products, labeled with greek indices $\alpha, \beta ...$, we have a master equation for each product:

$$\dot{n}_i^\alpha = \sum_j (C_{ij}^\alpha n_j^\alpha - C_{ji}^\alpha n_i^\alpha) \tag{1.3}$$

We also need a master equation for the evolution of the money of each person:

$$\dot{\$}_i = -\sum_{j,\alpha} (S_{ij}^\alpha n_j^\alpha - S_{ji}^\alpha n_i^\alpha), \tag{1.4}$$

where S_{ji}^α is the rate exchange of money between i and j. Again, conservation of each product separately and conservation of total money are fulfilled: $\sum_i \dot{n}_i^\alpha = 0$, $\sum_i \dot{\$}_i = 0$. From the point of view of the equations, the money variable \$ behaves as another product. If the prize for a product is fixed and do not depend on the consumer/seller, we can factorize it, writing

$$\dot{\$}_i = -\sum_\alpha p^\alpha \sum_j (C_{ij}^\alpha n_j^\alpha - C_{ji}^\alpha n_i^\alpha) = -\sum_\alpha p^\alpha \dot{n}_i^\alpha, \tag{1.5}$$

where p^α is the prize of each product. Otherwise, we always can write $S_{ij}^\alpha = p_j^\alpha C_{ij}^\alpha$, so j sells the product with a prize p_j^α. Finally, the prizes can also evolve over time, following a supply-demand law like:

$$\dot{p}^\alpha = -r^\alpha \sum_i \dot{n}_i^\alpha, \tag{1.6}$$

where r^α gives the rate of change of the prize over time per number of product. However, if the product is conserved, as we are considering, then $dotp_\alpha = 0$ and the prizes are conserved. I should use the reservoir quantity n_0 instead of the total number.

1.1 Reservoir model

Lets study the evolution of just one product for simplicity. Consider $N+1$ consumers/sellers, where the 0 guy is the "reservoir": all the purchases and sales are only realized through him. So the different consumers/sellers don't buy/sell among them but only with the reservoir. Therefore, the coefficients of the C matrix can be written as: $C_{i0} = c_i$, $C_{0i} = v_i$, $C_{ij} = 0$ for $i, j \neq 0$. Here, c_i and v_i stand for the rate of buying and selling of the i guy. The master equations take the form:

$$\begin{cases} \dot{n}_i &= c_i n_0 - v_i n_i, \\ \dot{n}_0 &= \sum_j (v_j n_j - c_j n_0) \end{cases} \tag{1.7}$$

Assuming that the reservoir es enough big and don't change appreciably over time (or equivalently, the rate of trading operations is enough fast), we have $\dot{n}_0 \simeq 0$, so, from second eq. of 1.7:

$$n_0 \simeq \frac{\sum_j v_j n_j}{\sum_j c_j}. \tag{1.8}$$

Putting this result into the first eq. of 1.7, we obtain

$$\dot{n}_i = \frac{c_i}{\sum_j c_j} \sum_j v_j n_j - v_i n_i = \xi_i \sum_j v_j n_j - v_i n_i = \xi_i \sum_{j \neq i} v_j n_j - v_i (1 - \xi_i) n_i, \tag{1.9}$$

where we have defined the competition factor $\xi_i = c_i / \sum_j c_j$, which takes values between 0 and 1. From the above equation, we can see that this model has the following properties:

- Notice that $\sum_i \xi_i = 1$ and therefore the above eq. fulfills $\sum_i \dot{n}_i = 0$.
- One of the advantages of this model is that we have reduced the number of parameters: from $N(N-1)$ parameters of the C_{ij} matrix, now we have $2N - 1$ of the ξ_i and v_i vectors (the -1 comes from the condition $\sum_i \xi_i = 1$).

- On the other hand, it accounts for the competition among consumers, through factor $\xi_i = c_i/\sum_j c_j$. The more i wants and can buy the product, the higher is c_i, and therefore ξ_i.
- The desire and capability to sell the product is parameterized in v_i. If i doesn't want to sell any unit, he can set $v_i = 0$.
- This model has a weak point: if i is able to buy almost all the product, leading to $\xi_i \simeq 1$, then he cannot sell a large amount of his product, since the sale term is proportional to $1-\xi_i \simeq 0$. However, this fact could be counteracted with an enough large v_i.
- Eq. 1.9 has the following formal solution:

$$n_i(t) = n_i(0)e^{-\int_0^t dt' v_i(t')(1-\xi_i(t'))} + \int_0^t dt' \xi_i(t') \sum_{j\neq i} v_j(t') n_j(t') e^{-\int_{t'}^t dt'' v_i(t'')(1-\xi_i(t''))}, \quad (1.10)$$

  where we have made explicit the temporal dependence.
- The evolution of money is given by $\dot{\$}_i = -p\dot{n}_i$, being stationary for the reservoir.
- A dynamical supply-demand law could be given by the number of units available in the reservoir: $\dot{p} = -rn_0 \simeq -r\sum_j v_j n_j/(\sum_j c_j)$.

2 Production and consumption

If we include to our modeling production and consumption of products, the master equations take the form

$$\dot{n}_i^\alpha = \sum_j (C_{ij}^\alpha n_j^\alpha - C_{ji}^\alpha n_i^\alpha) + \Lambda_i^\alpha - \Gamma_i^\alpha n_i^\alpha \quad (2.1)$$

where Λ_i^α is the production rate and Γ_i^α the consumption rate per unit of product. Analogously, the equation for the money is

$$\dot{\$}_i = -\sum_\alpha \left[p^\alpha \sum_j (C_{ij}^\alpha n_j^\alpha - C_{ji}^\alpha n_i^\alpha) + \lambda_i^\alpha \Lambda_i^\alpha \right] + J_i = \sum_\alpha [-p^\alpha(\dot{n}_i^\alpha + \Gamma_i^\alpha n_i^\alpha) + (p^\alpha - \lambda_i^\alpha)\Lambda_i^\alpha] + J_i, \quad (2.2)$$

being λ_i^α the prize that producing the product costs, and J_i a possible external source of money (e.g. a job not related with production or commerce). We can see that, if there is no consumption and the number of units remains stationary, there is a gain given by $(p^\alpha - \lambda_i^\alpha)$. The total number of products and money are not conserved anymore, only the market terms:

$$\sum_i \dot{n}_i^\alpha = \sum_i \Lambda_i^\alpha - \Gamma_i^\alpha n_i^\alpha; \quad \sum_i \dot{\$}_i = \sum_i \left[J_i - \sum_\alpha \lambda_i^\alpha \Lambda_i^\alpha \right] \quad (2.3)$$

Should we include a term $-J$ in the producer equation? CHECK

3 Simple relevant cases

Lets study some simple cases.

3.1 Producer-Consumer

We think in an very simple model of a producer which provides some product for the people, which is represented by an effective consumer. Making use of the reservoir model, we can write the master equations as

$$\begin{cases} \dot{n}_P &= \Lambda - V n_P, \\ \dot{n}_C &= V n_P - \Gamma n_C \end{cases} \tag{3.1}$$

$$\begin{cases} \dot{\$}_P &= -p\dot{n}_P + (p-\lambda)\Lambda = pVn_P - \lambda\Lambda, \\ \dot{\$}_C &= -p(\dot{n}_C + \Gamma n_C) + J = -pVn_P + J \end{cases} \tag{3.2}$$

In the stationary limit, when $\dot{n}_C = \dot{n}_P \simeq 0$ if C consumes all the product that C provides, we get that the leftover quantity that C keeps is

$$n_C \simeq \Lambda/\Gamma \tag{3.3}$$

In addition, if $\dot{\$}_C \simeq 0$, then $\dot{\$}_P = J - \lambda\Lambda$. P wins money depending on the difference $J - \lambda\Lambda$.

3.2 Labour theory of value

The labour theory of value can easily fit in this framework. Let's call n^{raw} some raw material or feedstock employed to make some product n^{prod}, and n^{work} the work force. Besides the producer P and the consumer P, now we have the worker W.

$$\begin{cases} \dot{n}_W^{work} &= -V^{work} n_W^{work}, \\ \dot{n}_P^{raw} &= V^{raw} n_0^{raw} - \Gamma_P^{raw} n_P^{raw}, \\ \dot{n}_P^{work} &= V^{work} n_W^{work} - \Gamma_P^{work} n_P^{work}, \\ \dot{n}_P^{prod} &= \Lambda - V^{prod} n_P^{prod}, \\ \dot{n}_C^{prod} &= V^{prod} n_P^{prod} - \Gamma_C^{prod} n_C^{prod} \end{cases} \tag{3.4}$$

$$\begin{cases} \dot{\$}_W &= -p^{work}\dot{n}_W = p^{work} V^{work} n_W^{work}, \\ \dot{\$}_P &= p^{prod} V^{prod} n_P^{prod} - \lambda\Lambda - p^{work} V^{work} n_W^{work} - p^{raw} V^{raw} n_0^{raw}, \\ \dot{\$}_C &= -p^{prod} V^{prod} n_P^{prod} \end{cases} \tag{3.5}$$

The labour theory of value states that

$$p^{prod} V^{prod} n_P^{prod} = \lambda\Lambda + p^{work} V^{work} n_W^{work} + p^{raw} V^{raw} n_0^{raw} \tag{3.6}$$

$\Lambda = \Lambda(n_P^{work}, n_P^{raw})$
CONTINUE

3.3 Two-producers competition

$$\begin{cases} \dot{n}_1 &= \Lambda_1 - V_1 n_1, \\ \dot{n}_2 &= \Lambda_2 - V_2 n_2, \\ \dot{n}_C &= V_1 n_1 + V_2 n_2 - \Gamma n_C \end{cases} \tag{3.7}$$

$$\begin{cases} \dot{\$}_1 &= p_1 V_1 n_1 - \lambda \Lambda_1, \\ \dot{\$}_1 &= p_2 V_2 n_2 - \lambda \Lambda_2, \\ \dot{\$}_C &= -p_1 V_1 n_1 - p_2 V_2 n_2 + J \end{cases} \tag{3.8}$$

V should be a dynamical parameter in order to allow that the consumer choose the less priced provider. Something like

$$V_1 - V_2 = -a(p_1 - p_2) \tag{3.9}$$

or

$$\dot{V}_i = -\sum_j (p_i - p_j) a_{ij} \tag{3.10}$$