Non inertial dynamics

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Abstract

A brief summary of main derivations and equations of dynamics in non inertial frames.

1 Time derivative in a non inertial frame

Let a coordinate frame called the body frame, moving arbitrarily with respect to the global inertial frame. The base vectors of the body frame $\boldsymbol{e}_i$ relate to those from the global frame $\boldsymbol{e}_i^g$ as

$$\boldsymbol{e}_i = \boldsymbol{R}\boldsymbol{e}_i^g \tag{1}$$

where $\boldsymbol{R}$ is the rotation matrix, which is orthogonal, i.e.

$$\boldsymbol{R}\boldsymbol{R}^T = \boldsymbol{I} \tag{2}$$

Let a vector $\boldsymbol{c}$. It can be written in terms of the body coordinate frame as

$$\boldsymbol{c} = \sum_i c_i \boldsymbol{e}_i \tag{3}$$

Its derivative

$$\dot{\boldsymbol{c}} = \sum_i \left(\dot{c}_i \boldsymbol{e}_i + c_i \dot{\boldsymbol{e}}_i\right) \tag{4}$$

Given that the global frame vectors are fixed, we can use Eq. 1 to write

$$\dot{\boldsymbol{c}} = \sum_i \left(\dot{c}_i \boldsymbol{e}_i + c_i \dot{\boldsymbol{R}}\boldsymbol{e}_i^g\right) \tag{5}$$

Since $\boldsymbol{R}$ is orthogonal, from Eq. 2 we can write

$$\dot{\boldsymbol{c}} = \sum_i \left(\dot{c}_i \boldsymbol{e}_i + c_i \dot{\boldsymbol{R}}\boldsymbol{R}^T\boldsymbol{R}\boldsymbol{e}_i^g\right) \tag{6}$$

Thus, defining the angular velocity matrix as

$$\boldsymbol{\Omega} = \dot{\boldsymbol{R}}\boldsymbol{R}^T \tag{7}$$

and hence

$$\dot{\boldsymbol{c}} = \sum_i \left(\dot{c}_i + c_i \boldsymbol{\Omega}\right)\boldsymbol{e}_i = \partial_t \boldsymbol{c} + \boldsymbol{\Omega}\boldsymbol{c} \tag{8}$$

where we use the notation $\partial_t \boldsymbol{c} = \sum_i \dot{c}_i \boldsymbol{e}_i$ to indicate that only the components of the vector (written in the body frame) are derived.

The angular velocity matrix $\boldsymbol{\Omega}$ is an antisymmetric matrix. That can be proven from the fact that the rotation matrix is orthogonal. Deriving Eq. 2:

$$0 = \dot{\boldsymbol{R}}\boldsymbol{R}^T + \boldsymbol{R}\dot{\boldsymbol{R}}^T \rightarrow \boldsymbol{\Omega} = -\boldsymbol{\Omega}^T \tag{9}$$

Since $\boldsymbol{\Omega}$ is an antisymmetric matrix, we can write its product to any vector as a cross product of a vector. This is because the components of any antisymmetric matrix can be written as proportional to the Levi Civita symbol $\Omega_{ij} \propto \epsilon_{ijk}$. Thus, we can define the angular velocity $\boldsymbol{\omega}$ vector as:

$$\boldsymbol{\omega} \times \boldsymbol{c} = \boldsymbol{\Omega c} \tag{10}$$

And finally:

$$\dot{\boldsymbol{c}} = \partial_t \boldsymbol{c} + \boldsymbol{\omega} \times \boldsymbol{c} \tag{11}$$

To get the components in the global frame, the $\boldsymbol{c}$ vector can be written as

$$\boldsymbol{c} = \sum_i c_i^g \boldsymbol{e}_i^g \tag{12}$$

Hence, equating from Eq. 3 and using Eq. 1, we get

$$c_i^g = R_{ij} c_j \tag{13}$$

2 Equations of motion

Let a particle with position $\boldsymbol{r}$ with respect to the body frame and velocity $\boldsymbol{v} = \dot{\boldsymbol{r}}$. We can write the equations of motion of the particle as

$$m\dot{\boldsymbol{v}} = \boldsymbol{F} \tag{14}$$

where $\boldsymbol{F}$ is the force. Expanding the left hand side using Eq. 11:

$$m \left(\partial_t \boldsymbol{v} + \boldsymbol{\omega} \times \boldsymbol{v}\right) = \boldsymbol{F} \tag{15}$$

In body frame components:

$$m \left(\dot{v}_i + \epsilon_{ijk} \omega_j v_k\right) = F_i \tag{16}$$

We can further expand the above equation to leave it in terms of the position vector. Using Eq. 11 again:

$$m \left(\partial_t^2 \boldsymbol{r} + \partial_t \boldsymbol{\omega} \times \boldsymbol{r} + 2\boldsymbol{\omega} \times \partial_t \boldsymbol{r} + \boldsymbol{\omega} \times \boldsymbol{\omega} \times \boldsymbol{r}\right) = \boldsymbol{F} \tag{17}$$

From the body frame, these terms in the acceleration are interpreted as fictitious forces:

1. Euler force: $\boldsymbol{F}_{Euler} = -m\partial_t \boldsymbol{\omega} \times \boldsymbol{r}$
2. Coriolis force: $\boldsymbol{F}_{Cor} = -m2\boldsymbol{\omega} \times \partial_t \boldsymbol{r}$
3. Centrifugal force: $\boldsymbol{F}_{Cent} = -m\boldsymbol{\omega} \times \boldsymbol{\omega} \times \boldsymbol{r}$

Similarly, for the moment equation, which relates the derivative of the angular moment $\boldsymbol{L} = \boldsymbol{I\omega}$ with the torque $\boldsymbol{M}$:

$$\boldsymbol{I}\dot{\boldsymbol{\omega}} = \boldsymbol{M} \tag{18}$$

and with Eq. 11

$$\boldsymbol{I}\partial_t \boldsymbol{\omega} + \boldsymbol{\omega} \times (\boldsymbol{I\omega}) = \boldsymbol{M} \tag{19}$$

The position of the particle in the global frame can be written as:

$$\boldsymbol{r}^g = \boldsymbol{Rr} + \boldsymbol{d} \tag{20}$$

where $\boldsymbol{d}$ is the displacement vector of the body frame with respect to the origin of the global frame.

In practice, for integration purposes, we want to integrate in the global reference frame from the acceleration local frame component $\boldsymbol{a} = \dot{\boldsymbol{v}}$, which from Eq. 13 is just

$$\ddot{x}_i^g = R_{ij} a_j \tag{21}$$

The above equations for 2D motion can be written as:

$$m \left(\dot{v}_x - \omega v_y\right) = F_x \tag{22}$$
$$m \left(\dot{v}_y + \omega v_x\right) = F_y \tag{23}$$
$$I_z \dot{\omega} = M_z \tag{24}$$