Random integers problem

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January 14, 2022

Let $R(x)$ be a random draw of integers between 0 and $x-1$ (inclusive). I repeatedly apply R, starting at 10^{100}. What's the expected number of repeated applications until I get zero?

One can frame the problem as a tower of states, where one can transition to lower states until reaching zero. Note that this is a Markov process, since the probability of reaching a given integer only depends on the previous state (the previous maximum integer x). Let $p_n(x|y)$ denote the transition probability of going from y to x in n steps. From an integer y we can only go downwards, e.g., to $x < y$. The mean of n steps required to go from x_0 to 0 is thus

$$E(n, x_0) = \sum_{i=1}^{x} i\, p_i(0|x_0) \tag{1}$$

Note that $p_1(y|x) = 1/x, \forall y < x$, since we have the same probability to go to any of the integers between 0 and x. Also, descending from an integer x in x steps implies perform x decays to the immediately inferior state, i.e., $p_x(0|x) = \prod_y^x p_1(y-1|y)$.

We can proceed in an inductive approach, starting from small initial integers.

- Firstly, for $x_0 = 1$, $p_1(0|1) = 1$, and hence $E(n, 1) = p_1(0|1) = 1$.
- For $x_0 = 2$, we have to consider $p_1(0|2) = 1/2$ and $p_2(0|2) = p_1(0|1)p_1(1|2) = 1 \times 1/2$. Thus, $E(n, 2) = p_1(0|2) + 2p_2(0|2) = 1/2 + 2 \times 1/2 = 1 + 1/2$.
- For $x_0 = 3$, $E(n, 3) = p_1(0|3)+2p_2(0|3)+3p_3(0|3)$. We have $p_3(0|3) = p_1(2|3)p_1(1|2)p_1(0|1) = (1/3)\times(1/2)\times 1$. On the other hand, to reach 0 in 2 steps, we have two possibilities, and hence we sum over both possible paths: $p_2(0|3) = p_1(2|3)p_1(0|2) + p_1(1|3)p_1(0|1) = 1/3 \times (1+1/2)$. Therefore, $E(n, 3) = 1 + 1/2 + 1/3$.

We can already see the trend

$$E(n, x) = \sum_{k=1}^{x} \frac{1}{k} = H_x, \tag{2}$$

where H_x is the x^{th} harmonic number. Since asymptotically, $H_x \sim \ln(x) + \gamma$, where γ is the Euler-Mascheroni constant, for a googol $x = 10^{100}$ we get $E(n, 10^{100}) = H_{10^{100}} \sim 230$.

Note also that

$$p_2(0|x) = \sum_{y=1}^{x-1} p_1(0|y)p_1(y|x) = \frac{1}{x}\sum_{y=1}^{x-1}\frac{1}{y} \tag{3}$$