Notes on the log-normal variance

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We want to obtain a proper expression for the variance $\sigma_\delta^2 = \langle \delta^2 \rangle = \langle (\Delta - 1)^2 \rangle$ using the log-normal PDF, enforcing that it should saturate to the linear variance σ_l^2 for high redshift, and taking into account the voids for later epochs.

In voids, the density decreases as

$$\rho = \bar{\rho}\Delta \propto t^{-3} \propto a^{-9/2} \tag{1}$$

Since in matter domination, the linear density goes as $\delta_l \propto a$, we can write for voids

$$\Delta \sim a^{-3/2} \sim \delta_l^{-3/2}. \tag{2}$$

Using the above equation, we can relate the linear variance σ_l^2 with the following moment of the distribution:

$$\langle \Delta^{-4/3} \rangle \sim \langle \delta_l^2 \rangle = \sigma_l^2 \tag{3}$$

The log-normal PDF is given by

$$P_{LG}(\Delta) = \frac{1}{\sqrt{2\pi\sigma^2}\Delta} exp\left[-\frac{(ln[\Delta/\mu])^2}{2\sigma_{LG}^2}\right], \tag{4}$$

where we choose $\mu = exp(-\sigma_{LG}^2/2)$ in order to fulfill the mass normalization. Then, σ_{LG} is the only free parameter in the PDF. With this distribution, it is possible to compute analytically the moments of the distribution as

$$\langle \Delta^n \rangle = exp[\frac{1}{2}n(n-1)\sigma_{LG}^2] \tag{5}$$

Therefore, we can compute the momentum of the PDF mentioned above as

$$\langle \Delta^{-4/3} \rangle = exp(\frac{14}{9}\sigma_{LG}^2). \tag{6}$$

Since $\langle \Delta^{-4/3} \rangle \sim \sigma_l^2$, we can write the parameter σ_{LG} as a function of the linear variance:

$$exp(\sigma_{LG}^2) = A\sigma_l^{9/7}, \tag{7}$$

where A is a proportionality constant. On the other hand, from Eq. 5, the variance can be written as

$$\sigma_\delta^2 = \langle \delta^2 \rangle = \langle (\Delta - 1)^2 \rangle = exp(\sigma_{LG}^2) - 1 = A\sigma_l^{9/7} - 1 \simeq A\sigma_l^{9/7} \tag{8}$$

when the last approximate equality stands as long as σ_l is larger than $\gtrsim 1$, what happens for low redshifts. Then, to get the proper limit at high redsfhits, we could write the variance for the log-normal PDF as

$$1 + \sigma_\delta = (1 + \frac{14}{9}\sigma_l)^{9/14}, \tag{9}$$

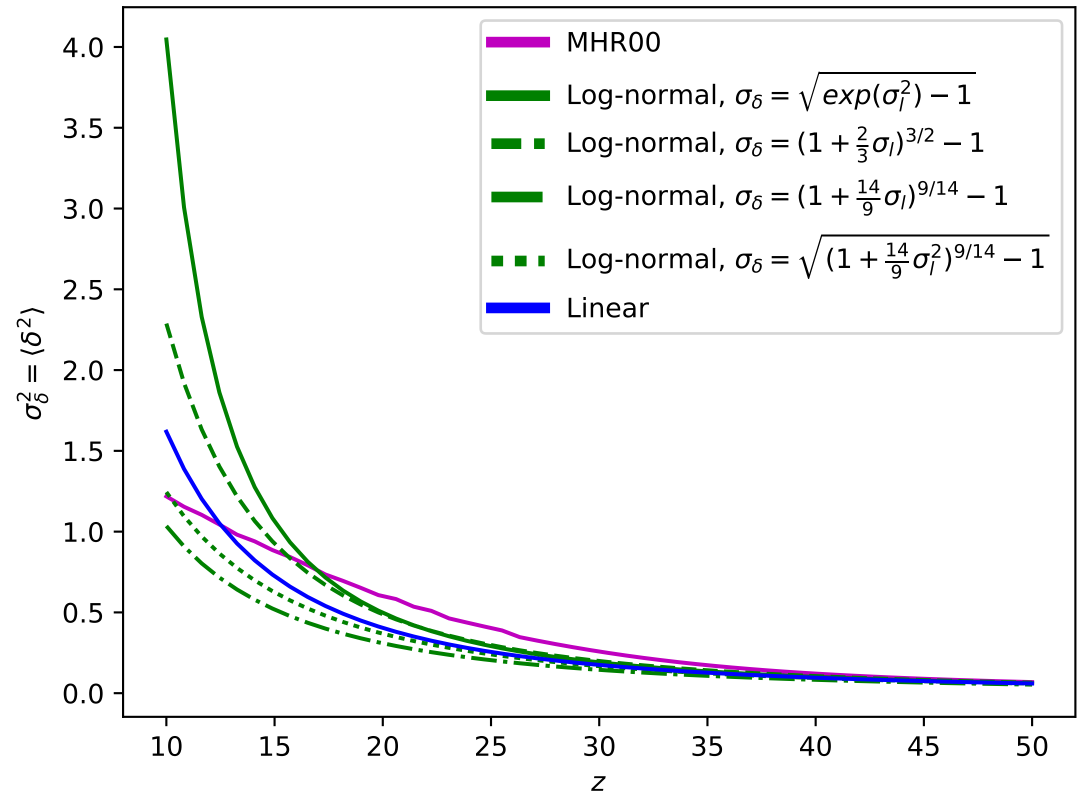

Figure 1: Comparison of the variance for the MHR00 and Log-normal cases.

which is plotted in Fig 1 with long dashed-dotted lines. The parameter of the PDF σ_{LG}^2 is written therefore in terms of the σ_δ. Other prescription which fulfills both limits would be

$$1 + \sigma_\delta^2 = (1 + \frac{14}{9}\sigma_l^2)^{9/14}, \tag{10}$$

depicted with dotted lines. The case with an exponent 3/2 is showed with dashed lines, together with the $\sigma_{LG} = \sigma_l$ in solid lines.